Warmup: Vector, scalar, or matrix.

1. $f:\mathbb{R}\to\mathbb{R}$ $\quad f'(x) =$ scalar
2. $f:\mathbb{R}^2\to\mathbb{R}^3$ $\quad f'(x) =$ matrix 3×2 $\begin{pmatrix}\nabla f_1\\ \nabla f_2\\ \nabla f_3\end{pmatrix}$
3. $g:\mathbb{R}^5\to\mathbb{R}$ $\quad \nabla g =$ vector $(g_{x_1}, g_{x_2}, \dots, g_{x_5})$
4. $g:\mathbb{R}^5\to\mathbb{R}$ $\quad$ (Direct. deriv of g in direction $(1,0,4,0,5)$ at $(1,-1,1,0,-1)$) = scalar
5. $g:\mathbb{R}^5\to\mathbb{R}$ $\quad g'(x) =$ matrix 1×5 (also vector) $\quad g'(x) = (\nabla g)$
6. $g:\mathbb{R}^5\to\mathbb{R}$ $\quad \frac{\partial g}{\partial x_3} =$ scalar

---

2.31 – Find pts of surface $\frac{z-y}{y^2+1} = \frac{-4}{x}$ where tangent plane $\parallel$ to $-2x+3y+z=12$.

Plan: $F(x,y,z) = \text{constant}$

$$\nabla F = c(-2,3,1)$$

$$\frac{z-y}{y^2+1} = \frac{-4}{x} \iff x(z-y) = -4(y^2+1)$$

$$F(x,y,z) = xz - xy + 4y^2 = -4$$

$$\nabla F = (z-y, -x+8y, x) = c(-2,3,1)$$

$$\left.\begin{aligned} z-y &= -2c \\ -x+8y &= 3c \\ \boxed{x = c} \end{aligned}\right\} \Rightarrow -c+8y = 3c$$

$$8y = 4c \Rightarrow \boxed{y = \frac{c}{2}}$$

$$z - \frac{c}{2} = -2c \Rightarrow \boxed{z = -\frac{3}{2}c}$$

$$c\left(-\tfrac{3}{2}c\right) - c\left(\tfrac{c}{2}\right) + 4\left(\tfrac{c}{2}\right)^2 = -4$$

$$-\tfrac{3}{2}c^2 - \tfrac{c^2}{2} + c^2 = -4$$

$$-c^2 = -4$$

$$c = \pm 2$$

$$\boxed{c=2 \quad (2, 1, -3) \qquad c=-2 \quad (-2, -1, 3)}$$

Upcoming theme - max's & mins.

Recall: Calc 1:

To find local max's, min's, saddle pts. of $f(x)$

- find cr. pts $f'(x) = 0$ (or undefined)

  solve for x.

- type of cr pt. $x = a$ cr. pt.

Taylor polynomial — near $x = a$,

$$f(x) \approx \underbrace{f(a) + \boxed{f'(a)(x-a)}}_{\text{tangent line}} + \frac{f''(a)}{2}(x-a)^2$$

with $f'(a) = 0$.

If $f''(a) > 0$, $f''(a) < 0$

saddle $f''(a) = 0$

To find global max's, mins on interval

$[a,b]$, $(a,b]$, (a,∞) etc.

- $f'(x) = 0$ to search for interior extrema (extrema = max's & min)
- look at what happens when $x \to$ endpts. (take limit or plug in)

• Compare values of $f(x)$ at those points you have found to get global max & min.

Multivariable Calculus: $f(\overset{x_1,\ x_2,}{x, y, \ldots})$ $\quad f: \mathbb{R}^n \to \mathbb{R}$

To find local max's, min's, saddle pts:

• Find critical pts $\nabla f = (0, 0, \ldots)$

solve for $(x, y, \ldots)$

• Taylor polynomial near a critical pt

$(x_1, x_2, \ldots) = (a_1, a_2, \ldots)$

dot product.

$$f(x_1, x_2, \ldots) \approx \underbrace{f(a_1, a_2, \ldots) + \overset{(0,0,\ldots)}{\nabla f(a_1, a_2, \ldots)} \cdot \underset{(x_1 - a_1,\ x_2 - a_2,\ \ldots)}{(x - a)}}_{\text{tangent hyperplane}}$$

$$+ \frac{1}{2!}(f_{11})(x_1 - a_1)^2 + \frac{1}{2!}(f_{12})(x_1 - a_1)(x_2 - a_2)$$

$$+ \frac{1}{2!}(f_{21})(x_2 - a_2)(x_1 - a_1) + \frac{1}{2!}(f_{22})(x_2 - a_2)^2$$

$$f_{11} = \frac{\partial^2 f}{\partial x_1^2}(a_1, a_2, \ldots), \quad f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}(a_1, a_2, \ldots), \text{ etc.}$$

$$f(x) \approx \underbrace{f(a) + \frac{1}{2} f_{11}(x_1 - a_1)^2 + f_{12}(x_1 - a_1)(x_2 - a_2) \cdots + \frac{f_{22}}{2}(x_2 - a_2)^2 + \cdots}_{\text{closest degree 2 polynomial that fits the (hyper-)surface.}}$$

slicker way of writing this: matrix of 2^{nd} derivs

$$f(x) \approx f(a) + \frac{1}{2}\left(x_1-a_1, x_2-a_2, \ldots\right)\begin{pmatrix} f_{11} & f_{12} & \cdot \\ f_{21} & f_{22} & \vdots \\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix} x_1-a_1 \\ x_2-a_2 \\ \vdots \end{pmatrix}$$

$1\times n \qquad n \times n \qquad n \times 1$

2 dimensions (x,y)

$$f(x,y) \approx f(\overset{\text{cr pt}}{a_1,a_2}) + \frac{1}{2}\left(x-a_1, y-a_2\right)\overset{\text{Hessian}}{\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}}\begin{pmatrix} x-a_1 \\ y-a_2 \end{pmatrix}$$

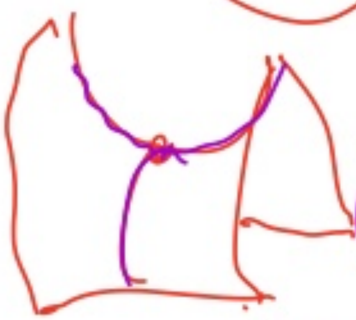
2^{nd} derivatives evaluated at $x=a_1$, $y=a_2$.

$\lambda_1, \lambda_2, \ldots \lambda_n$
Eigenvalues of Hessian:

all eigenvalues >0

local min

all eigenvalues are negative
local max



some eigenvalues >0
some <0

If we have zero eigenvalue(s) — not completely sure of the shape.
$\lambda_i = 0$

Finding global maxes & mins in multivariable calculus $f(x,y,\ldots)$ on Domain

- Find critical pts $\nabla f = (0,0,\dots)$ to find possible extrema in the interior of the domain.
- Restrict the function to the boundary to find any extrema on the boundary (might need to take limit if boundary is not included.)

First skill we need: Eigenvalues of a square matrix.
(this will help us calculate the eigenvalues of the Hessian → type of critical pt.)

Given a square matrix, eg.

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 8 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ to find the eigenvalues } \lambda,$$

1. subtract λ from diagonal entries
2. $\det(A-\lambda I)=0$ (Take the determinant, set to 0)
3. Solve for solutions λ ←

$$(A-\lambda I) = \begin{pmatrix} 2-\lambda & 1 & 3 \\ 8 & -2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

bottom row

$$\det(A-\lambda I) = |A-\lambda I| = 0\begin{vmatrix} 1 & 3 \\ -2-\lambda & 0 \end{vmatrix} - 0\begin{vmatrix} 2-\lambda & 3 \\ 8 & 0 \end{vmatrix}$$

$$+ (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 8 & (-2-\lambda) \end{vmatrix}$$

$$= 0 + 0 + (1-\lambda)\left[(2-\lambda)(-2-\lambda) - 8\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-4+\lambda^2-8\right] = 0$$

$$(1-\lambda)(\lambda^2 - 12) = 0$$

$$\boxed{\lambda = 1 \text{ or } \lambda = \sqrt{12} \text{ or } \lambda = -\sqrt{12}}$$

2 eigenvalues +, 1 eigenvalue −.

Ex Find the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) - 4$$

$$= \lambda^2 - 8\lambda + 7 - 4$$

$$= \lambda^2 - 8\lambda + 3 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-12}}{2}$$

$$= \frac{8 \pm \sqrt{52}}{2} \quad (\text{both} > 0!)$$

Calculus example.

Find and classify the critical points of $f(x,y) = 3x^2 - 2x^3 - 3y^2 + 6xy - 4$.

$$\nabla f = (\underbrace{6x - 6x^2 + 6y}_{f_x}, \underbrace{-6y + 6x}_{f_y}) = (0,0)$$

(After dividing by 6)

$$x - x^2 + y = 0$$
$$-y + x = 0 \Rightarrow y = x$$

$$x - x^2 + x = 0$$
$$-x^2 + 2x = 0$$
$$x(-x+2) = 0 \Rightarrow x = 0 = y \text{ or } x = 2 = y$$

Two critical pts: $(0,0)$ & $(2,2)$.

Hessian: $f_{xx} = 6 - 12x$

$f_{yx} = f_{xy} = 6$

$f_{yy} = -6$

$$\begin{pmatrix} 6-12x & 6 \\ 6 & -6 \end{pmatrix}$$

$$(0,0) \Rightarrow \text{Hessian} = \begin{pmatrix} 6 & 6 \\ 6 & -6 \end{pmatrix}$$

$$\begin{vmatrix} 6-\lambda & 6 \\ 6 & -6-\lambda \end{vmatrix} = (6-\lambda)(-6-\lambda) - 36$$

$$= -36 + \lambda^2 - 36 = \lambda^2 - 72$$

$$\lambda = \pm\sqrt{72}$$

one +, one −

$\Rightarrow$ $\boxed{(0,0) \text{ is a saddle pt.}}$

$$(2,2) \Rightarrow \text{Hessian} = \begin{pmatrix} -18 & 6 \\ 6 & -6 \end{pmatrix}$$

$$\begin{vmatrix} -18-\lambda & 6 \\ 6 & -6-\lambda \end{vmatrix} = \underbrace{(-18-\lambda)(-6-\lambda)}_{(18+\lambda)(6+\lambda)} - 36$$

$$= \lambda^2 + 24\lambda + 108 - 36$$

$$\Rightarrow = \lambda^2 + 24\lambda + 72$$

$$\lambda = \frac{-24 \pm \sqrt{24^2 - 4\cdot 72}}{2}$$

both negative

$\Rightarrow$ $\boxed{(2,2) \text{ is a local max}}$